Theorems on Cardinal Numbers.

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In this paper it is proved ($\star$ 2) that for cardinal numbers (finite or infinite) if $\alpha < \alpha'$ and $\beta < \beta'$, then $\alpha + \beta < \alpha' + \beta'$. It is already known (cf. $\star$ 4·37 in my article on *Cardinal Numbers*, in this Journal, Vol. XXIV, 1902) that in this case $\alpha + \beta \leqq \alpha' + \beta'$. Another form of $\star$ 2 is $\star$ 1, which is the most useful form for deductions. By the help of $\star$ 1, it is proved ($\star$ 3 and $\star$ 4) that if n be a finite number and α_0 (as usual) the first infinite number, and if $n \times \beta = \alpha_0 \times \beta$, then $\beta = \alpha_0 \times \beta$. It is already known (cf. Zermelo, Gott. Nachricht., 1901) that if $\beta = n \times \beta$, then $\beta = \alpha_0 \times \beta$; and also it is obvious that if $\beta = \alpha_0 \times \beta$, then $\beta = n \times \beta$: then the present theorem completes this set of results.

Furthermore it is proved ($\star$ 6) that if α is of the form $\alpha_0 \times \gamma$ [i. e. an $N\alpha_0$, cf. *Cardinal Numbers*, $\star$ 30] and $\beta < \alpha$, then $\beta + \sigma = \alpha$ implies $\sigma = \alpha$. This proposition was assumed without proof in my article on *Cardinal Numbers* (cf. $\star$ 16·22, for example). It is already known that $\beta + \alpha = \alpha$.

It will be noticed that a condition attaches itself to the hypothesis of $\star$ 1, and thence to the hypotheses of the succeeding theorems, namely (stating it generally), that any two numbers less than the numbers considered, are such that one of them is either greater than, or equal to, or less than the other. I am not aware of any proof that one of these relations must hold for any two cardinal numbers; but classes of infinite cardinal numbers are known for which this condition is true.

The proofs are written in Peano's notation explained in the article on *Cardinal Numbers*, already cited; except that here $\subset$ is used for "is contained in," as applied to classes, and $\supset$ is used for "implies," as applied to propositions, instead of $\supset$ for both these ideas. Also, in the proofs where a hypothesis is made to hold for the remainder of the proof "constr" for "constructive" is written after it.

★1 $\alpha, \beta, \alpha', \beta' \varepsilon Nc . \alpha + \beta = \alpha' + \beta' : \mu \leqq \alpha . \nu \leqq \beta . \supset_{\mu, \nu} . \mu \lesseqgtr \nu : \supset : \alpha \geqq \alpha' . \vee . \beta \geqq \beta',$

$[R \varepsilon 1 \to 1 . u \cap v = \Lambda . u \cup v = \rho . u' \cap v' = \Lambda . u' \cup v' = \breve{\rho},$ constr (1)

$S \varepsilon 1 \to 1 . \sigma \subset u \cap \rho v' . \breve{\sigma} = v \cap \rho u',$ constr (2)

$p = (u \cup v) - (\sigma \cup \breve{\sigma}) .$ constr (3)

$R' = R_p \cup SR \cup \breve{S}R,$ constr (4)

$(4) . \supset . R' \varepsilon 1 \to 1 . \breve{\rho}' u = \breve{\rho}(u - \sigma) \cup \breve{\rho}(v \cap \rho u'),$ (5)

$(5) . \supset . u' \subset \breve{\rho}' u . \supset . Nc ' u' \leqq Nc ' u,$ (6)

$(2) . (6) . \supset : Nc ' (u \cap \rho v') \geqq Nc ' (v \cap \rho u') . \supset . Nc ' u' \leqq Nc ' u .$ (7)

Similarly $Nc ' (v \cap \rho u') \geqq Nc ' (u \cap \rho v') . \supset . Nc ' v' \leqq Nc ' v,$ (8)

$(7) . (8) : \mu \leqq \alpha . \nu \leqq \beta . \supset_{\mu, \nu} . \mu \lesseqgtr \nu : u \varepsilon \alpha . v \varepsilon \beta . u' \varepsilon \alpha' . v' \varepsilon \beta' : \supset :$ Prop],

★2 $\alpha, \beta, \alpha', \beta' \varepsilon Nc . \alpha < \alpha' . \beta < \beta' : \mu \leqq \alpha . \nu \leqq \beta . \supset_{\mu, \nu} . \mu \gtreqless \nu : \supset . \alpha + \beta < \alpha' + \beta',$

[★1 . $\supset$. Prop.]

★3 $m, n \varepsilon Nc$ fin $- \iota 0 . m \geqq n . \beta \varepsilon Nc : \mu < \beta . \nu < \beta . \supset_{\mu, \nu} \mu \gtreqless \nu : (m + n) \times \beta = \alpha_0 \times \beta : \supset . m \times \beta = \alpha_0 \times \beta .$

[★1 . Hyp . $\supset : m \times \beta + n \times \beta = \alpha_0 \times \beta + \alpha_0 \times \beta . \supset :$

$m \times \beta \geqq \alpha_0 \times \beta . \vee . n \times \beta \geqq \alpha_0 \times \beta ,$ (1)

$l \varepsilon Nc$ fin $. l \times \beta \geqq \alpha_0 \times \beta . \supset . l \times \beta = \alpha_0 \times \beta,$ (2)

$m \geqq n . \supset : n \times \beta = \alpha_0 \times \beta . \supset . m \times \beta = \alpha_0 \times \beta,$ (3)

(1) . (2) . (3) . $\supset$. Prop].

★4 $n \varepsilon Nc$ fin $- \iota 0 . \beta \varepsilon Nc : \mu < \beta . \nu < \beta . \supset_{\mu, \nu} . \mu \gtreqless \nu : n \times \beta = \alpha_0 \times \beta : \supset . \beta = \alpha_0 \times \beta .$

[★3 . $\supset$. Prop.]

★5 $m, n \varepsilon Nc$ fin $- \iota 0 . m \sigma' n . \beta \varepsilon Nc : \mu < \beta . \nu < \beta . \supset_{\mu, \nu} . \mu \gtreqless \nu : m \times \beta = n \times \beta : \supset . \beta = \alpha_0 \times \beta,$

$[m \times \beta = n \times \beta . \supset . m \times \beta = \alpha_0 \times \beta .$ (1)

Note: This theorem (1) is given by Zermelo, Gott. Nachrichten, 1901.

(1) . ★4 . $\supset$. Prop.]

★6 $\alpha \varepsilon N\alpha_0 : \mu \leqq \alpha . \nu \leqq \alpha . \supset_{\mu, \nu} . \mu \gtreqless \nu : \beta < \alpha . \beta + \sigma = \alpha : \supset : \sigma = \alpha,$

$[\text{Hyp} . \beta + \sigma = \alpha . \supset . \beta + \sigma = \alpha + \alpha,$ (1)

$(1) . ★1 . \beta < \alpha . \supset . \sigma \geqq \alpha,$ (2)

$\beta + \sigma = \alpha . \supset . \sigma \leqq \alpha,$ (3)

(2) . (3) . $\supset$. Prop.]

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